4

Matrices

4.1 DEFINITION

Let us consider a set of simultaneous equations,

$$x + 2y + 3z + 5t = 0$$
$$4x + 2y + 5z + 7t = 0$$
$$3x + 4y + 2z + 6t = 0.$$

Now we write down the coefficients of x, y, z, t of the above equations and enclose them within brackets and then we get

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 4 & 2 & 5 & 7 \\ 3 & 4 & 2 & 6 \end{bmatrix}$$

The above system of numbers, arranged in a rectangular array in rows and columns and bounded by the brackets, is called a matrix.

It has got 3 rows and 4 columns and in all $3 \times 4 = 12$ elements. It is termed as 3×4 matrix, to be read as [3 by 4 matrix]. In the double subscripts of an element, the first subscript determines the row and the second subscript determines the column in which the element lies, a_{ij} lies in the ith row and jth column.

4.2 VARIOUS TYPES OF MATRICES

(1) **Row Matrix.** If a matrix has only one row and any number of columns, it is called a Row matrix, *e.g.*,

$$[2 \ 7 \ 3 \ 9]$$

(2) **Column Matrix.** A matrix, having one column and any number of rows, is called a Column matrix, e.g., $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(3) **Null Matrix or Zero Matrix.** Any matrix, in which all the elements are zeros, is called a Zero matrix or Null matrix *e.g.*,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(*d*) **Square Matrix.** A matrix, in which the number of rows is equal to the number of columns, is called a square matrix *e.g.*,

$$\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$$

(*e*) **Diagonal Matrix.** A square matrix is called a diagonal matrix, if all its non-diagonal elements are zero *e.g.*,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(*f*) **Unit or Identity Matrix.** A square matrix is called a unit matrix if all the diagonal elements are unity and non-diagonal elements are zero *e.g.*,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(*g*) **Symmetric Matrix.** A square matrix will be called symmetric, if for all values of i and j, *i.e.* $a_{ij} = a_{ji}$ or $A' = A$ *e.g.*,

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

(*h*) **Skew Symmetric Matrix.** A square matrix is called skew symmetric matrix, if

(1) $a_{ij} = -a_{ji}$ for all values of i and j, or $A' = -A$

(2) All diagonal elements are zero, *e.g.*,

$$\begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$$

(*i*) **Triangular Matrix.** A square matrix, all of whose elements below the leading diagonal are zero, is called an *upper triangular matrix*. A square matrix, all of whose elements above the leading diagonal are zero, is called a lower triangular matrix *e.g.*,

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 6 & 7 \end{bmatrix}$$

Upper triangular matrix Lower triangular matrix

(*j*) **Transpose of a Matrix.** If in a given matrix A, we interchange the rows and the corresponding columns, the new matrix obtained is called the transpose of the matrix A and is denoted by A' or A^T *e.g.*,

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \\ 6 & 7 & 8 \end{bmatrix}, \quad A' = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 0 & 7 \\ 4 & 5 & 8 \end{bmatrix}$$

(*k*) **Orthogonal Matrix.** A square matrix A is called an orthogonal matrix if the product of the matrix A and the transpose matrix A' is an identity matrix *e.g.*,

$$A \,.\, A' = I$$

if $|A| = 1$, matrix A is proper.

(*l*) **Conjugate of a Matrix**

Let $$A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$$

Conjugate of matrix of A is $\overline{A}$

$$\therefore \quad \overline{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}$$

(*m*) **Matrix A^θ.** Transpose of the conjugate of a matrix A is denoted by A^θ.

Let
$$A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$$
$$\bar{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & +i & 3+2i \end{bmatrix}$$
$$(\bar{A})' = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}$$
or
$$A^\theta = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}$$

(*n*) **Unitary Matrix.** A square matrix A is said to be unitary if
$$A^\theta A = I$$

e.g. $$A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}, \quad A^\theta = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}, \quad A \cdot A^\theta = I$$

(*o*) **Hermitian Matrix.** A square matrix $A = (a_{ij})$ is called **Hermitian matrix, if every** *i-j*th element of A is equal to conjugate complex *j-i*th element of A.

In other words $a_{ij} = \bar{a}_{ji}$

e.g. $$\begin{bmatrix} 1 & 2+3i & 3+i \\ 2-3i & 2 & 1-2i \\ 3-i & 1+2i & 5 \end{bmatrix}$$

Necessary and sufficient condition for a matrix A to be Hermitian is that $A = A^\theta$ *i.e.* conjugate transpose of A

or $$A = (\bar{A})'.$$

(*p*) **Skew Hermitian Matrix.** A square matrix $A = (a_{ij})$ will be called a Skew Hermitian matrix if every *i-j*th element of A is equal to negative conjugate complex of *j-i*th element of A.

In other words $a_{ij} = -\bar{a}_{ji}$

All the elements in the principal diagonal will be of the form
$$a_{ii} = -\bar{a}_{ii} \quad \text{or} \quad a_{ii} + \bar{a}_{ii} = 0$$

If $a_{ii} = a + ib$ then $\bar{a}_{ii} = a - ib$

$$(a + ib) + (a - ib) = 0 \quad \text{or} \quad 2a = 0 \text{ or } a = 0$$

So a_{ii} is pure imaginary or $a_{ii} = 0$.

Hence all the diagonal elements of a Skew Hermitian Matrix are either zeros or pure imaginary.

e.g. $$\begin{bmatrix} i & 2-3i & 4+5i \\ -(2+3i) & 0 & 2i \\ -(4-5i) & 2i & -3i \end{bmatrix}$$

The necessary and sufficient condition for a matrix A to be Skew Hermitian is that
$$A^\theta = -A$$
$$(\bar{A})' = -A$$

(*q*) **Idempotent Matrix.** A matrix, such that $A^2 = A$ is called Idempotent Matrix.

e.g. $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, $A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$

(*r*) **Periodic Matrix.** A matrix A will be called a Periodic Matrix, if

$$A^{k+1} = A$$

where k is a +ve integer. If k is the least +ve integer, for which $A^{k+1} = A$, then k is said to be the period of A. If we choose $k = 1$ we get $A^2 = A$ and we call it to be idempotent matrix.

(*s*) **Nilpotent Matrix.** A matrix will be called a Nilpotent matrix, if $A^k = 0$ **(null matrix)** where k is a +ve integer ; if however k is the least +ve integer for which $A^k = 0$, then k is the *index* of the nilpotent matrix.

e.g., $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$, $A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

A is nilpotent matrix whose index is 2.

(*t*) **Involuntary Matrix.** A matrix A will be called an Involuntary matrix, if $A^2 = I$ (unit matrix). Since $I^2 = I$ always $\therefore$ Unit matrix is involuntary.

(*u*) **Equal Matrices.** Two matrices are said to be equal if

(*i*) They are of the same order.

(*ii*) The elements in the corresponding positions are equal.

Thus if $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Here $A = B$

Example 1. *Find the value of x, y, z and 'a' which satisfy the matrix equation.*

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$$

Solution. As the given matrices are equal, so their corresponding elements are equal.

$$x + 3 = 0 \quad \text{or} \quad x = -3 \qquad \ldots(i)$$

$$2y + x = -7 \qquad \ldots(ii)$$

$$z - 1 = 3 \quad \text{or} \quad z = 4 \qquad \ldots(iii)$$

$$4a - 6 = 2a \text{ or } a = 3 \qquad \ldots(iv)$$

Putting the value of $x = -3$ from (i) into (ii) we have

$$2y - 3 = -7 \text{ or } y = -2$$

$$x = -3, \quad y = -2, \quad z = 4, \quad a = 3 \qquad \textbf{Ans.}$$

4.3 ADDITION OF MATRICES

If A and B be two matrices of the same order, then their sum, $A + B$ is defined as the matrix, each element of which is the sum of the corresponding elements of A and B.

Thus if $A = \begin{bmatrix} 4 & 2 & 5 \\ 1 & 3 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}$

then $$A + B = \begin{bmatrix} 4+1 & 2+0 & 5+2 \\ 1+3 & 3+1 & -6+4 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 7 \\ 4 & 4 & -2 \end{bmatrix}$$

If $A = [a_{ij}]$, $B = [b_{ij}]$ then $A + B = [a_{ij} + b_{ij}]$

Example 2. *Show that any square matrix can be expressed as the sum of two matrices, one symmetric and the other anti-symmetric.*

Solution. Let A be a given square matrix.

Then $$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

Now $$(A + A')' = A' + A = A + A'.$$

$\therefore$ $A + A'$ is a symmetric matrix.

Also $$(A - A')' = A' - A = -(A - A')$$

$\therefore$ $A - A'$ or $\frac{1}{2}(A - A')$ is an anti-symmetric matrix.

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

= Symmetric matrix + antisymmetric matrix **Proved.**

Example 3. *Write matrix (A) given below as the sum of a symmetric and a skew symmetric matrix.*

$$A = \begin{pmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{pmatrix}$$

Solution: $A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix}$ On transposing we get $A' = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 5 & 6 \\ 4 & 3 & 3 \end{bmatrix}$

On adding A and A', we have

$$A + A' = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 & -1 \\ 2 & 5 & 6 \\ 4 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 10 & 9 \\ 3 & 9 & 6 \end{bmatrix} \quad ...(1)$$

On subtracting A' from A we get

$$A - A' = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -2 & -1 \\ 2 & 5 & 6 \\ 4 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 5 \\ -4 & 0 & -3 \\ -5 & 3 & 0 \end{bmatrix} \quad ...(2)$$

On adding (1) and (2), we have

$$2A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 10 & 9 \\ 3 & 9 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 5 \\ -4 & 0 & -3 \\ -5 & 3 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 5 & \frac{9}{2} \\ \frac{3}{2} & \frac{9}{2} & 3 \end{bmatrix} + \begin{bmatrix} 0 & 2 & \frac{5}{2} \\ -2 & 0 & -\frac{3}{2} \\ \frac{-5}{2} & \frac{3}{2} & 0 \end{bmatrix}$$

= [Symmetric matrix] + [Skew symmetric matrix.] **Ans.**

4.4. PROPERTIES OF MATRIX ADDITION

Only matrices of the same order can be added or subtracted.

(*i*) **Commutative Law.** $A + B = B + A$.

(*ii*) **Associative law.** $A + (B + C) = (A + B) + C$.

4.5 SUBTRACTION OF MATRICES

The difference of two matrices is a matrix, each element of which is obtained by subtracting the elements of the second matrix from the corresponding element of the first.

$$A - B = [a_{ij} - b_{ij}]$$

Thus
$$\begin{bmatrix} 8 & 6 & 4 \\ 1 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 5 & 1 \\ 7 & 6 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 8-3 & 6-5 & 4-1 \\ 1-7 & 2-6 & 0-2 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 3 \\ -6 & -4 & -2 \end{bmatrix}$$ **Ans.**

4.6 SCALAR MULTIPLE OF A MATRIX

If a matrix is multiplied by a scalar quantity k, then each element is multiplied by k, *i.e.*

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix}$$

$$3A = 3\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 3\times2 & 3\times3 & 3\times4 \\ 3\times4 & 3\times5 & 3\times6 \\ 3\times6 & 3\times7 & 3\times9 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 12 \\ 12 & 15 & 18 \\ 18 & 21 & 27 \end{bmatrix}$$

Exercise 4.1

1. (*a*) If $A = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$, represent it as $A = B + C$ where B is a symmetric and C is a skew-symmetric matrix.

(*b*) Express $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 7 & 1 \\ 5 & 9 & 3 \end{bmatrix}$ as a sum of symmetric and skew-symmetric matrix.

Ans. (*a*) $A = \begin{bmatrix} -1 & \frac{9}{2} & 3 \\ \frac{9}{2} & 3 & 2 \\ 3 & 2 & \frac{9}{2} \end{bmatrix} + \begin{bmatrix} 0 & \frac{5}{2} & -2 \\ \frac{-5}{2} & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$ (*b*) $A = \begin{bmatrix} 1 & \frac{5}{2} & \frac{5}{2} \\ \frac{5}{2} & 7 & 5 \\ \frac{5}{2} & 5 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & 0 & -4 \\ \frac{5}{2} & 4 & 0 \end{bmatrix}$

2. Matrices A and B are such that

$$3A - 2B = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \text{ and } -4A + B = \begin{pmatrix} -1 & 2 \\ -4 & 3 \end{pmatrix}$$

Find A and B.

Ans. $A = \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 \\ 4 & -1 \end{bmatrix}$

3. Given $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$

Find x, y, z and w.

Ans. $x = 2, y = 4, z = 1, w = 3$

4. If $A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Find (*i*) $2A + 3B$ (*ii*) $3A - 4B$.

Ans. (*i*) $= \begin{bmatrix} 3 & 10 & 3 \\ 8 & 3 & 6 \\ 2 & 2 & 13 \end{bmatrix}$, (*ii*) $= \begin{bmatrix} -4 & -2 & -4 \\ -5 & -4 & 9 \\ 3 & 3 & -6 \end{bmatrix}$

5. If A is a skew-symmetric matrix of odd order, then the determinant of A is

(*a*) -1. (*b*) 0. (*c*) 1. (*d*) a real number. **Ans.** (*b*)

4.7 MULTIPLICATION

The product of two matrices A and B is only possible if the number of columns in A is equal to the number of rows in B.

Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be an $n \times p$ matrix. Then the product AB of these matrices is an $m \times p$ matrix $C = [c_{ij}]$ where

$$c_{ij} = a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + a_{i3}\, b_{3j} + ... + a_{in}\, b_{nj}$$

4.8 $(AB)' = B'A'$

If A and B are two matrices conformal for product AB, then show that $(AB)' = B'A'$, where dash represents transpose of a matrix.

Solution. Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{ij})$ be $n \times p$ matrix.

Since AB is an $m \times p$ matrix, $(AB)'$ is a $p \times m$ matrix.

Further B' is $p \times n$ matrix and A' an $n \times m$ matrix and therefore $B'A'$ is a $p \times m$ matrix.

Thus $(AB)'$ and $B'A'$ are matrices of the same order.

Now the (j, i)th element of $(AB)'$ = (i, j)th element of $(AB) = \sum_{k=1}^{n} a_{ik} b_{kj}$ ...(1)

Also the jth row of B' is $b_{1j}, b_{2j} .... b_{jn}$, and ith column of A' is $a_{i1}, a_{i2}, a_{i3} .... a_{in}$.

$\therefore$ (j, i)th element of $B'A' = \sum_{k=1}^{n} b_{kj}\, a_{ik}$ ...(2)

From (1) and (2) we have (j, i)th element of $(AB)'$ = (j, i)th element of $B'A'$.

As the matrices $(AB)'$ and $B'A'$ are of the same order and their corresponding elements are equal, we have $(AB)' = B'A'$. **Proved**

4.9 PROPERTIES OF MATRIX MULTIPLICATION

1. Multiplication of matrices is not commutative.

$$AB \neq BA$$

2. Matrix multiplication is associative, if confirmability is assured.

$$A\,(BC) = (AB)\,C$$

3. Matrix multiplication is distributive with respect to addition.

$$A\,(B + C) = AB + AC$$

4. Multiplication of matrix A by unit matrix.

$$AI = IA = A$$

5. Multiplicative inverse of a matrix exists if $|A| \neq 0$.

$$A \,.\, A^{-1} = A^{-1} \,.\, A = I$$

Example 4. *If* $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$

obtain the product AB and explain why BA is not defined.

Solution. The number of columns in A is 3 and the number of rows in B is also 3, therefore the product AB is defined.

$$AB = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \times \begin{bmatrix} C_1 & C_2 \\ 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} R_1\,C_1 & R_1\,C_2 \\ R_2\,C_1 & R_2\,C_2 \\ R_3\,C_1 & R_3\,C_2 \end{bmatrix}$$

R_1, R_2, R_3 are rows of A and C_1, C_2 are columns of B.

$$\begin{bmatrix} \boxed{0\ 1\ 2} & \boxed{\begin{matrix} 1 \\ -1 \\ 2 \end{matrix}} & \boxed{0\ 1\ 2} & \boxed{\begin{matrix} -2 \\ 0 \\ -1 \end{matrix}} \\ \boxed{1\ 2\ 3} & \boxed{\begin{matrix} 1 \\ -1 \\ 2 \end{matrix}} & \boxed{1\ 2\ 3} & \boxed{\begin{matrix} -2 \\ 0 \\ -1 \end{matrix}} \\ \boxed{2\ 3\ 4} & \boxed{\begin{matrix} 1 \\ -1 \\ 2 \end{matrix}} & \boxed{2\ 3\ 4} & \boxed{\begin{matrix} -2 \\ 0 \\ -1 \end{matrix}} \end{bmatrix}$$

For convenience of multiplication, we write the columns in horizontal rectangles.

$$\begin{bmatrix} \boxed{0\quad 1\quad 2} & \boxed{0\quad 1\quad 2} \\ \boxed{1\quad -1\quad 2} & \boxed{-2\quad 0\quad -1} \\ \boxed{1\quad 2\quad 3} & \boxed{1\quad 2\quad 3} \\ \boxed{1\quad -1\quad 2} & \boxed{-2\quad 0\quad -1} \\ \boxed{2\quad 3\quad 4} & \boxed{2\quad 3\quad 4} \\ \boxed{1\quad -1\quad 2} & \boxed{-2\quad 0\quad -1} \end{bmatrix}$$

$$= \begin{bmatrix} 0\times 1+1\times(-1)+2\times 2 & 0\times -2+1\times 0+2\times(-1) \\ 1\times 1+2\times(-1)+3\times 2 & 1\times -2+2\times 0+3\times(-1) \\ 2\times 1+3\times(-1)+4\times 2 & 2\times -2+3\times 0+4\times(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 0-1+4 & 0+0-2 \\ 1-2+6 & -2+0-3 \\ 2-3+8 & -4+0-4 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$$ **Ans.**

Since the number of columns of B is (2) $\neq$ the number of rows of A is 3, BA is not defined.

Example 5. *If* $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ *(A.M.I.E., Winter 1999)*

form the products AB and BA, and show that $AB \neq BA$.

Solution.

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-0+3 & 0-2+6 & 2-4+0 \\ 2+0-1 & 0+3-2 & 4+6-0 \\ -3+0+2 & 0+1+4 & -6+2+0 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 1 & 1 & 10 \\ -1 & 5 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0-6 & -2+0+2 & 3-0+4 \\ 0+2-6 & 0+3+2 & 0-1+4 \\ 1+4+0 & -2+6+0 & 3-2+0 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 7 \\ -4 & 5 & 3 \\ 5 & 4 & 1 \end{bmatrix}$$

$\therefore$ $AB \neq BA$ **Proved**

Example 6. If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ *and* $C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$

Verify that $(AB)C = A(BC)$ *and* $A(B+C) = AB + AC$.

Solution. We have

$$AB = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(2)+(2)(2) & (1)(1)+(2)(3) \\ (-2)(2)+(3)(2) & (-2)(1)+(3)(3) \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix}$$

$$BC = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -6+2 & 2+0 \\ -6+6 & 2+0 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 0 & 2 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \times \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -3+4 & 1+0 \\ 6+6 & -2+0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 12 & -2 \end{bmatrix}$$

$$B + C = \begin{bmatrix} 2+(-3) & 1+1 \\ 2+2 & 3+0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix}$$

(*i*) $(AB)C = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix} \times \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -18+14 & 6+0 \\ -6+14 & 2+0 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ 8 & 2 \end{bmatrix}$ ...(1)

and $A(BC) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -4 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -4+0 & 2+4 \\ 8+0 & -4+6 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ 8 & 2 \end{bmatrix}$ ...(2)

Thus from (1) and (2), we get

$$(AB)C = A(BC)$$

(*ii*) $A(B+C) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -1+8 & 2+6 \\ 2+12 & -4+9 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 14 & 5 \end{bmatrix} \quad \ldots(3)$$

$$AB + AC = \begin{bmatrix} 6+1 & 7+1 \\ 2+12 & 7-2 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 14 & 5 \end{bmatrix} \quad \ldots(4)$$

Thus from (3) and (4), we get

$$A(B+C) = AB + AC \qquad \textbf{Verified}$$

Example 7. *If* $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ *show that* $A^2 - 4A - 5I = 0$ *where* I, 0 *are the unit matrix and the null matrix of order 3 respectively. Use this result to find* A^{-1}.

Solution.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^2 - 4A - 5I = 0 \qquad \text{or} \qquad 5I = A^2 - 4A$$

By multiplying by A^{-1}, we get

$$5A^{-1} = A - 4I$$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} \qquad \textbf{Ans.}$$

Example 8. *If* $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

verify that $(AB)' = B'A'$, ($'$) *indicates transpose.*

Solution. $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+0 & 0+2-1 & 0+0-3 \\ 3+0+0 & 0+0+2 & 0+0+6 \\ 4+10+0 & 0+5+0 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -3 \\ 3 & 2 & 6 \\ 14 & 5 & 0 \end{bmatrix} \quad ...(1)$$

$$(AB)' = \begin{bmatrix} 5 & 3 & 14 \\ 1 & 2 & 5 \\ -3 & 6 & 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix}, \quad B' = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+0 & 3+0+0 & 4+10+0 \\ 0+2-1 & 0+0+2 & 0+5+0 \\ 0+0-3 & 0+0+6 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 14 \\ 1 & 2 & 5 \\ -3 & 6 & 0 \end{bmatrix} \quad ...(2)$$

Proved.

From (1) and (2), $(AB)' = B'A'$

Example 9. $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & 2 \end{bmatrix}$.

Verify the result $(A+B)^2 = A^2 + AB + BA + B^2$.

Solution. We have

$$A+B = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1+3 & 2+(-1) & -1+1 \\ 2+0 & 0+0 & 3+2 \\ 0+4 & 1+(-3) & 2+2 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 5 \\ 4 & -2 & 4 \end{bmatrix}$$

Then

$$(A+B)^2 = (A+B)(A+B) = \begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 5 \\ 4 & -2 & 4 \end{bmatrix}\begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 5 \\ 4 & -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (4)(4)+(1)(2)+(0)(4) & (4)(1)+(1)(0)+(0)(-2) & (4)(0)+(1)(5)+(0)(4) \\ (2)(4)+(0)(2)+(5)(4) & (2)(1)+(0)(0)+(5)(-2) & (2)(0)+(0)(5)+(5)(4) \\ (4)(4)+(-2)(2)+(4)(4) & (4)(1)+(-2)(0)+(4)(-2) & (4)(0)+(-2)(5)+(4)(4) \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 4 & 5 \\ 28 & -8 & 20 \\ 28 & -4 & 6 \end{bmatrix}$$

Now

$$A^2 = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+0 & 2+0-1 & -1+6-2 \\ 2+0+0 & 4+0+3 & -2+0+6 \\ 0+2+0 & 0+0+2 & 0+3+4 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 3 \\ 2 & 7 & 4 \\ 2 & 2 & 7 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 3+0-4 & -1+0+3 & 1+4-2 \\ 6+0+12 & -2+0-9 & 2+0+6 \\ 0+0+8 & 0+0-6 & 0+2+4 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 3 \\ 18 & -11 & 8 \\ 8 & -6 & 6 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3-2+0 & 6+0+1 & -3-3+2 \\ 0+0+0 & 0+0+2 & 0+0+4 \\ 4-6+0 & +8+0+2 & -4-9+4 \end{bmatrix} = \begin{bmatrix} 1 & 7 & -4 \\ 0 & 2 & 4 \\ -2 & 10 & -9 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 9+0+4 & -3+0-3 & 3-2+2 \\ 0+0+8 & 0+0-6 & 0+0+4 \\ 12-0+8 & -4+0-6 & 4-6+4 \end{bmatrix} = \begin{bmatrix} 13 & -6 & 3 \\ 8 & -6 & 4 \\ 20 & -10 & 2 \end{bmatrix}$$

$$\therefore \quad A^2+AB+BA+B^2 = \begin{bmatrix} 5 & 1 & 3 \\ 2 & 7 & 4 \\ 2 & 2 & 7 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 3 \\ 18 & -11 & 8 \\ 8 & -6 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 7 & -4 \\ 0 & 2 & 4 \\ -2 & 10 & -9 \end{bmatrix} + \begin{bmatrix} 13 & -6 & 3 \\ 8 & -6 & 4 \\ 20 & -10 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5-1+1+13 & 1+2+7-6 & 3+3-4+3 \\ 2+18+0+8 & 7-11+2-6 & 4+8+4+4 \\ 2+8-2+20 & 2-6+10-10 & 7+6-9+2 \end{bmatrix} = \begin{bmatrix} 18 & 4 & 5 \\ 28 & -8 & 20 \\ 28 & -4 & 6 \end{bmatrix}$$

Thus $A^2+AB+BA+B^2 = \begin{bmatrix} 18 & 4 & 5 \\ 28 & -8 & 20 \\ 28 & -4 & 6 \end{bmatrix} = (A+B)^2$. **Proved.**

Example 10. *Show by means of an example that in matrices $AB = 0$ does not necessarily mean that either $A = 0$ or $B = 0$, where 0 stands for the null matrix.*

Solution. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$

$$AB = \begin{bmatrix} 1-2+1 & 2-4+2 & 3-6+3 \\ -3+4-1 & -6+8-2 & -9+12-3 \\ -2+2+0 & -4+4+0 & -6+6+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad AB = 0$

But here neither $A = 0$ nor $B = 0$ **Proved.**

Example 11. *If $AB = AC$, it is not necessarily true that $B = C$ i.e. like ordinary algebra, the equal matrices in the identity cannot be cancelled.*

Solution. Thus $AB = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}$

$$AC = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}$$

Hence $AB = AC$. But $B \neq C$.

Example 12. *Represent each of the transformations* $x_1 = 3y_1 + 2y_2$, $y_1 = z_1 + 2z_2$ *and* $x_2 = -y_1 + 4y_2$, $y_2 = 3z_1$ *by the use of matrices and find the composite transformation which expresses* x_1, x_2 *in terms of* z_1, z_2.

Solution. The equations in the matrix form are

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \ldots (1)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \ldots (2)$$

Substituting the values of y_1, y_2 in (2) we get

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 9z_1 + 6z_2 \\ 11z_1 - 2z_2 \end{bmatrix}$$

$x_1 = 9z_1 + 6z_2, \quad x_2 = 11z_1 - 2z_2$ **Ans.**

Example 13. *If A, B and C are matrices such that the products AC and BC obey the commutative law, prove that*

$$(AB + BA)\,C = C\,(AB + BA)$$

Solution.

$$\begin{aligned}
(AB + BA)\,C &= (AB)\,C + (BA)\,C && \text{Distributive law} \\
&= A\,(BC) + B\,(AC) && \text{Associative law} \\
&= A\,(CB) + B\,(CA) && BC \text{ and } AC \text{ obey commutative law} \\
&= (AC)\,B + (BC)\,A && \text{Associative law} \\
&= (CA)\,B + (CB)\,A && \text{Commutative law} \\
&= C\,(AB) + C\,(BA) && \text{Associative law} \\
&= C\,[AB + BA] && \text{Distributive law}
\end{aligned}$$

Proved

Example 14. *Prove that the product of two matrices*

$$\begin{bmatrix} \cos^2\theta & \cos\theta \sin\theta \\ \cos\theta \sin\theta & \sin^2\theta \end{bmatrix} \text{ and } \begin{bmatrix} \cos^2\phi & \cos\phi \sin\phi \\ \cos\phi \sin\phi & \sin^2\phi \end{bmatrix}$$

is zero when θ *and* ϕ *differ by an odd multiply of* $\frac{\pi}{2}$.

Solution.

$$\begin{bmatrix} \cos^2\theta & \cos\theta \sin\theta \\ \cos\theta \sin\theta & \sin^2\theta \end{bmatrix} \times \begin{bmatrix} \cos^2\phi & \cos\phi \sin\phi \\ \cos\phi \sin\phi & \sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta \cos^2\phi + \cos\theta \sin\theta \cos\phi \sin\phi & \cos^2\theta \cos\phi \sin\phi + \cos\theta \sin\theta \sin^2\phi \\ \cos\theta \sin\theta \cos^2\phi + \sin^2\theta \cos\phi \sin\phi & \cos\theta \sin\theta \cos\phi \sin\phi + \sin^2\theta \sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta \cos\phi (\cos\theta \cos\phi + \sin\theta \sin\phi) & \cos\theta \sin\phi (\cos\theta \cos\phi + \sin\theta \sin\phi) \\ \sin\theta \cos\phi (\cos\theta \cos\phi + \sin\theta \sin\phi) & \sin\theta \sin\phi (\cos\theta \cos\phi + \sin\theta \sin\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta \cos\phi \cos(\theta - \phi) & \cos\theta \sin\phi \cos(\theta - \phi) \\ \sin\theta \cos\phi \cos(\theta - \phi) & \sin\theta \sin\phi \cos(\theta - \phi) \end{bmatrix}.$$

$$\theta - \phi = (2n + 1)\frac{\pi}{2}$$

Given

$$\cos(\theta - \phi) = \cos(2n + 1)\frac{\pi}{2} = 0$$

$\therefore$ The product $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$ **Proved.**

Example 15. *Verify that*

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \text{ is orthogonal.}$$

Solution.

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \quad \therefore \quad A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$AA' = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence A is an orthogonal matrix. **Verified.**

Example 16. *Determine the values of α, β, γ when*

$$\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \text{ is orthogonal.}$$

Solution. Let $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$

On transposing A, we have

$$A' = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$$

If A is orthogonal, then $AA' = I$

$$\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}\begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} 4\beta^2+\gamma^2 & 2\beta^2-\gamma^2 & -2\beta^2+\gamma^2 \\ 2\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 \\ -2\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we have

$$\left.\begin{aligned} 4\beta^2+\gamma^2 &= 1 \\ 2\beta^2-\gamma^2 &= 0 \end{aligned}\right\} \Rightarrow \beta = \pm\frac{1}{\sqrt{6}}, \gamma = \pm\frac{1}{\sqrt{3}}$$

But $\alpha^2+\beta^2+\gamma^2 = 1$ as $\beta = \pm\frac{1}{\sqrt{6}}, \gamma = \pm\frac{1}{\sqrt{3}}, \alpha = \pm\frac{1}{\sqrt{2}}$ **Ans.**

Example 17. *Prove that* $(AB)^n = A^n \cdot B^n$ *if* $A \cdot B = B \cdot A$

Solution.

$$(AB)^1 = AB = (A) \cdot (B)$$

$$(AB)^2 = (AB) \cdot (AB) = (ABA) \cdot B = \{A(AB)\} \cdot B$$

$$= (A^2B) \cdot B = A^2 \cdot (B \cdot B) = A^2 \cdot B^2$$

Suppose that

$$(AB)^n = A^n \cdot B^n$$

$$(AB)^{n+1} = (AB)^n \cdot (AB) = (A^n \cdot B^n) \cdot (AB) = A^n \cdot (B^n A) \cdot B$$

$$= A^n \cdot (B^{n-1} \cdot BA) \cdot B = A^n \cdot (B^{n-1} \cdot AB) \cdot B$$

$$= A^n \cdot (B^{n-2} \cdot B \cdot AB) \cdot B = A^n \cdot (B^{n-2} \cdot AB \cdot B) \cdot B$$

$$= A^n \cdot (B^{n-2} \cdot AB^2) \cdot B, \text{ continuing the process } n \text{ times.}$$

$$= A^n \cdot (A \cdot B^n) \cdot B = A^n \cdot (A \cdot B^{n+1}) = A^{n+1} \cdot B^{n+1}$$

Hence taking the above to be true for $n = n$, we have shown that it is true for $n = n + 1$ and also it was true for $n = 1, 2, \ldots$ so it is universally true. **Proved.**

Exercise 4.2

1. Compute AB, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 6 & 4 \\ 4 & 7 & 5 \end{bmatrix}$$

Ans. $\begin{bmatrix} 20 & 38 & 26 \\ 47 & 92 & 62 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$. Form the product AB and BA. Show that $AB \neq BA$.

3. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(*i*) Calculate AB and BA. Hence evaluate $A^2B + B^2A$.

(*ii*) Show that for any number k, $(A + kB^2)^3 = kI$, where I is the unit matrix. (*A.M.I.E., Winter 1999*)

4. If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ choose α and β so that $(\alpha I + \beta A)^2 = A$

Ans. $\alpha = \beta = \pm \frac{1}{\sqrt{2}}$

5. If $A = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $B = [-1 \quad 1 \quad 0]$ and $C = \begin{bmatrix} 3 & 0 \\ -2 & 1 \\ 0 & 1 \end{bmatrix}$,

verify that $(ABC)^T = C^T B^T A^T$, where T denotes the transpose.

6. Write the following transformation in matrix form:

$$x_1 = \frac{\sqrt{3}}{2} y_1 + \frac{1}{2} y_2 \; ; \; x_2 = -\frac{1}{2} y_1 + \frac{\sqrt{3}}{2} y_2$$

Hence, find the transformation in matrix form which expresses y_1, y_2 in terms of x_1, x_2.

Ans. $y_1 = \frac{\sqrt{3}}{2} x_1 - \frac{1}{2} x_2, \quad y_2 = \frac{1}{2} x_1 + \frac{\sqrt{3}}{2} x_2$

7. If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is a unit matrix, show that $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

8. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then show that $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$, where n is positive integer.

9. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then show that $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

10. If $f(x) = x^3 - 20x + 8$, find $f(A)$ where $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ **Ans.** 0

11. Show that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}^{-1}$

12. $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ show that $A^3 = A^{-1}$.

13. Verify whether the matrix $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$ is orthogonal.

14. Verify that $\frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{bmatrix}$ is an orthogonal matrix.

15. Show that $\begin{bmatrix} \cos\phi & 0 & \sin\phi \\ \sin\theta\sin\phi & \cos\theta & -\sin\theta\cos\phi \\ -\cos\theta\sin\phi & \sin\theta & \cos\theta\cos\phi \end{bmatrix}$ is an orthogonal matrix.

16. Show that $A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ is an orthogonal matrix.

17. If A and B are square matrices of the same order, explain in general
(*i*) $(A+B)^2 \neq A^2 + 2AB + B^2$ (*ii*) $(A-B)^2 \neq A^2 - 2AB + B^2$ (*iii*) $(A+B)(A-B) \neq A^2 - B^2$

18. Let A and B be any two matrices such that $AB = 0$ and A is non-singular.
Then (*a*) $B = 0$; (*b*) B is also non-singular; (*c*) $B = A$; (*d*) B is singular.
(*A.M.I.E.T.E., Winter 1996*) **Ans.** (*d*)

4.10 ADJOINT OF A SQUARE MATRIX

Let the determinant of the square matrix A be $|A|$.

If $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ Then $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

The matrix formed by the co-factors of the elements in $|A|$ is

$$\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$